Achievement Standard 2.3

Calculus

Calculus involves the processes of <u>differentiation</u> and <u>integration</u>

Differentiation

The derivitive of x^n is nx^{n-1}

Function	Derivitive
x^4	$4x^3$
x^5	$5x^4$
$3x^2$	$6x$
$6x^3$	$18x^2$
$x^5 - 3x^3$	$5x^4 - 9x^2$

The derivitive of $5x$ is 5
$8x$ is 8
$-x$ is -1

The derivitive of a constant term is 0.

The derivitive of a function $f(x)$ is written $f'(x)$

eg Differentiate the following

1) $f(x) = 3x^2 - 8x + 5$
$f'(x) = 6x - 8 + 0$

2) $f(x) = x^4 + 2x^3 - 4x^2 + x - 3$
$f'(x) = 4x^3 + 6x^2 - 8x + 1$

3) $g(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 2x + 11$
$g'(x) = x^2 + 5x - 2$

Second Derivative

The second derivative is written $f''(x)$

eg 1) $f(x) = x^3 + 5x^2 - 3x + 12$
$f'(x) = 3x^2 + 10x - 3$
$f''(x) = 6x + 10$

2) $f(x) = \frac{3}{4}x^4 - x^3 + \frac{5}{2}x^2 - x + 2$
$f'(x) = 3x^3 - 3x^2 + 5x - 1$
$f''(x) = 9x^2 - 6x + 5$

Harder differentiation

1) $f(x) = 8\sqrt{x}$
$f(x) = 8x^{1/2}$
$f'(x) = 4x^{-1/2}$ or $\frac{4}{\sqrt{x}}$

2) $f(x) = 12\sqrt[3]{x}$
$f(x) = 12x^{1/3}$
$f'(x) = 4x^{-2/3}$ or $\frac{4}{\sqrt[3]{x^2}}$

3) $f(x) = 8x - \frac{5}{x}$
$f(x) = 8x - 5x^{-1}$
$f'(x) = 8 + 5x^{-2}$
$= 8 + \frac{5}{x^2}$

4) $f(x) = 5x + \frac{3}{x^2}$
$f(x) = 5x + 3x^{-2}$
$f'(x) = 5 - 6x^{-3}$
$= 5 - \frac{6}{x^3}$

A new notation

If the equation is written $y = f(x)$, then the derivative is written $\frac{dy}{dx}$

eg 1) $y = 3x^2 - 4x + 5$

$\frac{dy}{dx} = 6x - 4$

2) $S = t^3 + 2t^2 - 3t + 4$

$\frac{ds}{dt} = 3t^2 + 4t - 3$

3) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

The second derivative is written $\frac{d^2y}{dx^2}$

4) $P = 5t^2 + 3t + 1$

$\frac{dP}{dt} = 10t + 3$

$\frac{d^2P}{dt^2} = 10$

5) $y = 3x^4 - x^2 + 7$

$\frac{dy}{dx} = 12x^3 - 2x$

$\frac{d^2y}{dx^2} = 36x^2 - 2$

Gradient or slope

The slope of a line is given by the derivative

Derivative = Slope

1) Find the slope of the curve
$f(x) = 3x^2 - 8x + 1$ when $x = 3$
$f(x) = 6x - 8$
Slope $= 6(3) - 8$
$= 10$

2) Find the slope of the curve
$y = x^2 + 5x - 2$ at the point $(4, 34)$
$\frac{dy}{dx} = 2x + 5$

Slope $= 2(4) + 5$
$= 13$

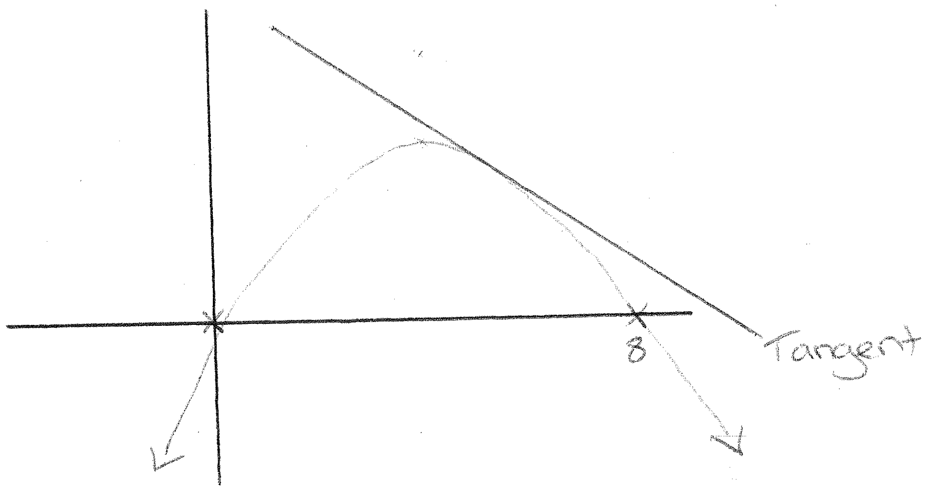
3) Find the co-ordinates of the point on the curve $y = x^2 - 2x - 5$, where the gradient is 6.
$\frac{dy}{dx} = 2x - 2$

Slope $= 2x - 2$
$6 = 2x - 2$
$8 = 2x$
$x = 4$

$y = 4^2 - 2(4) - 5$
$y = 3$
Point is $(4, 3)$

Tangent to curve

Find the equation of the tangent to the curve $y = x(8-x)$ at the point $(6,12)$



$y = 8x - x^2$

$\frac{dy}{dx} = 8 - 2x$

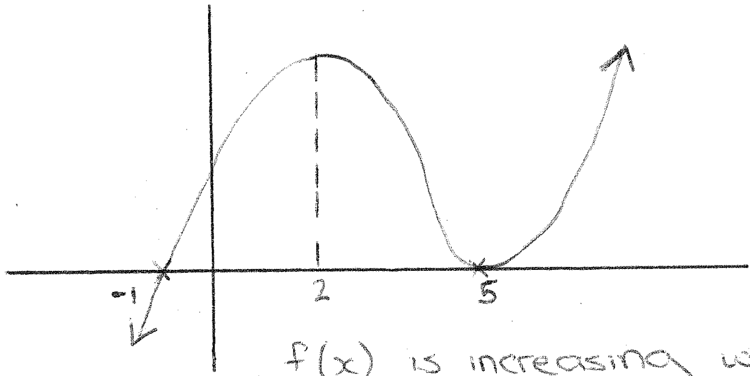
Slope $= 8 - 2(6) = {}^-4$

Using
$$y - y_1 = m(x - x_1)$$
$$y - 12 = {}^-4(x-6)$$
$$y - 12 = {}^-4x + 24$$
$$y = {}^-4x + 36$$
$$(\text{or } 4x + y - 36 = 0)$$

Increasing and decreasing functions

A function $f(x)$ is increasing if its slope is positive.
i.e. $f(x) > 0$

A function $f(x)$ is decreasing if its slope is negative
i.e. $f(x) < 0$



$f(x)$ is increasing when $x < 2$ and when $x > 5$
$f(x)$ is decreasing when $2 < x < 5$

Example: Find where the graph of $y = x^2 - 4x + 5$ is a decreasing function.

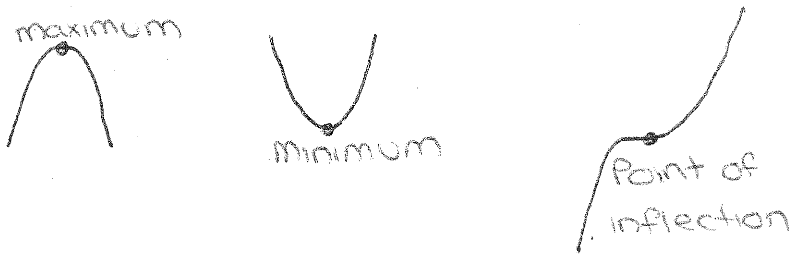
$$\frac{dy}{dx} = 2x - 4$$

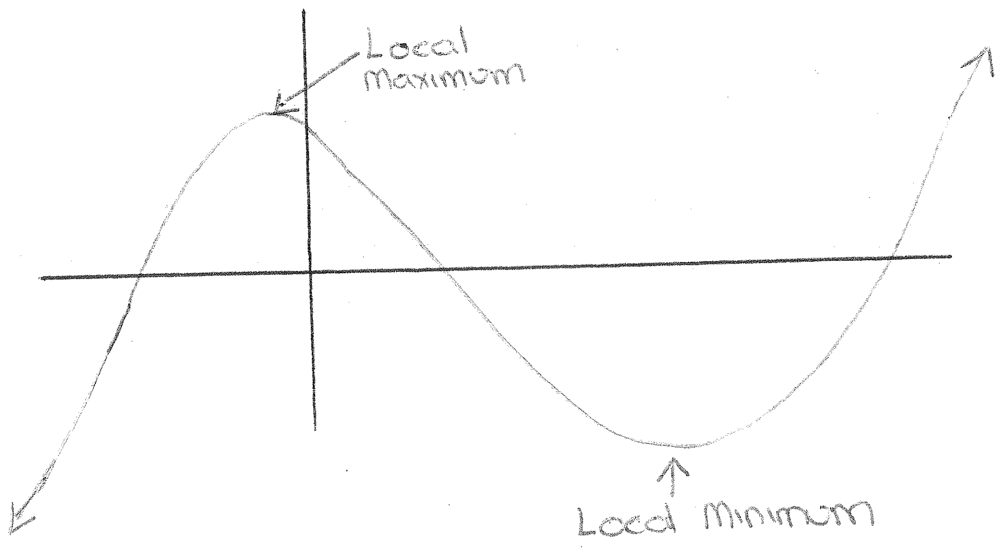
Require
$$2x - 4 < 0$$
$$2x < 4$$
$$x < 2$$

Decreasing when $x < 2$

Turning Points (Stationary Points)

There are three types of stationary points, maximum or minimum or "point of inflection"



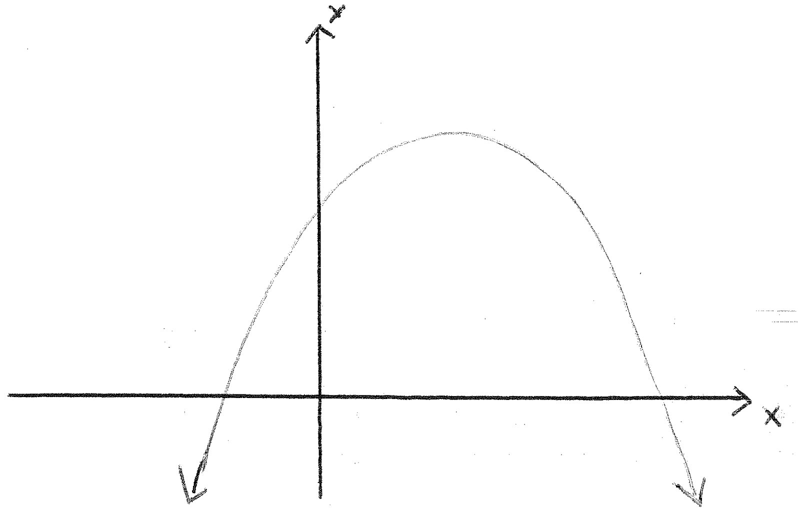


At a turning point the slope $=0$
ie $\underline{f'(x) = 0}$

At a turning point the slope = 0
ie $f'(x) = 0$

If the turning point is a maximum $f''(x) < 0$
If the turning point is a minimum $f''(x) > 0$

Examples <u>i)</u> The graph shows $y = -2x^2 + 6x + 3$



Find the coordinates of the T.P.
Show that the T.P. is a maximum

Differentiate: $\frac{dy}{dx} = -4x + 6$

Put derivative = 0: $-4x + 6 = 0$

$-4x = -6$

$x = 1.5$

Substitute for x: $y = -2(1.5)^2 + 6(1.5) + 3$

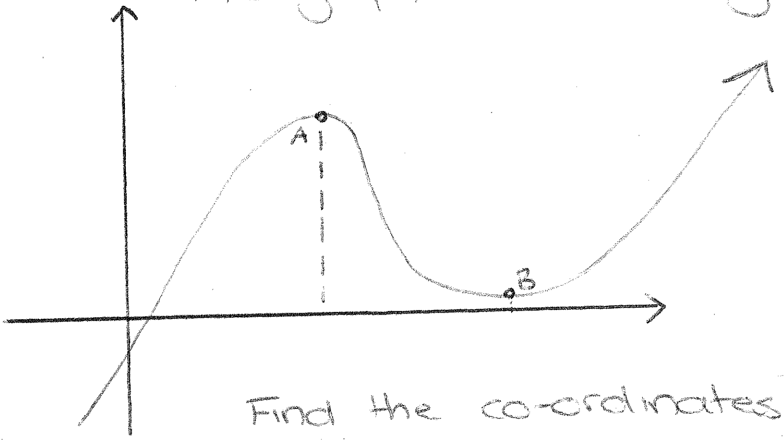
$y = 7.5$

T.P. at $(1.5, 7.5)$

$\frac{d^2y}{dx^2} = -4$

since $\frac{d^2y}{dx^2} < 0$ it is a max

2) The graph shows $y = x^3 - 9x^2 + 24x - 8$



Find the co-ordinates of A and B

$\frac{dy}{dx} = 3x^2 - 18x + 24$

$3x^2 - 18x + 24 = 0$

$3(x - 6x + 8) = 0$

$3(x-4)(x-2) = 0$

$x \in \{2, 4\}$

When $x = 2$

$y = (2)^3 - 9(2)^2 + 24(2) - 8$

$y = 12$

When $x = 4$

$y = (4)^3 - 9(4)^2 + 24(4) - 8$

$y = 8$

Turning points at $A(2,12)$ and $B(4,8)$

$\frac{d^2y}{dx^2} = 6x - 18$

when $x = 2$, $\frac{d^2y}{dx^2} = -6$

So T.P. at $x = 2$ is a max

When $x = 4$, $\frac{d^2y}{dx^2} = 6$

So T.P. at $x = 4$ is a min.

Optimisation Problems

1) A plane flies from Dunedin to Christchurch. Its height h (metres) after t minutes is given by $h = -5.5t^2 + 330t + 50$. Find out when the plane reaches its maximum and what this height is.

$$\therefore \frac{dh}{dt} = -11t + 330$$

$$= -11t + 330 = 0$$

$$330 = 11t$$

$$t = 30 \text{ minutes}$$

$$\therefore h\,max = -5.5(30)^2 + 330(30) + 50$$

$$= 5000m$$

2) The cost per km of running a fishing boat depends on its velocity v (knots) at which you travel.

$$C = v^2 - 24v + 150$$

Find the speed v which minimises the cost and cost per km.

$$\therefore \frac{dc}{dv} = 2v - 24$$

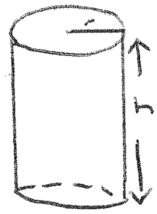
$$2v - 24 = 0$$

$$2v = 24$$

$$v = 12 \text{ knots}$$

$$\therefore C_{min} = (12)^2 - 24(12) + 150$$

$$= \$6 \text{ per km}$$

3) 

The surface area of a cylinder is given by
$S = 2\pi r^2 + 2\pi r h$
The volume V of a cylinder is given
by $V = \pi$

Integration

Integration is the opposite of differentiation
The symbol $\int$ means "the integral"

eg $\int 2x \cdot dx$

the integral   of 2x   with respect to x

The rule for integration is

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$$

c is called the "constant of integration"

Example:

1) $\int x^3 \cdot dx = \frac{x^4}{4} + C$

2) $\int 6x^2 \cdot dx = \frac{6x^3}{3} + C$
$= 2x^3 + C$

3) $\int (x^2 - 5x + 4) \cdot dx$
$= \frac{x^3}{3} - \frac{5x^2}{2} + 4x + C$

Evaluating the Constant

1) Find $f(x)$ if $f'(x) = 4x-2$
and $f(5)=32$

$$f(x) = \int (4x-2)\,dx$$
$$= \frac{4x^2}{2} - 2x + C$$
$$= 2x^2 - 2x + C$$

$$32 = 2(5)^2 - 2(5) + C$$
$$32 = 50 - 10 + C$$
$$32 = 40 + C$$
$$C = -8$$

$$\therefore f(x) = 2x^2 - 2x - 8$$

2) A curve has gradient function

$$\frac{dy}{dx} = 8x^3 - 2x + 3$$

The curve passes through the point $(1,5)$
Find the equation of the curve

$$\int (8x^3 - 2x + 3)\,dx$$
$$y = \frac{8x^4}{4} - x^2 + 3x + C$$
$$y = 2x^4 - x^2 + 3x + C$$

$$5 = 2(1)^4 - (1)^2 + 3(1) + C$$
$$5 = 2 - 1 + 3 + C$$
$$5 = 4 + C$$
$$C = 1$$

Equation of curve is $y = 2x^4 - x^2 + 3x + 1$

Definite Integrals

$$\int_a^b f'(x).dx = f(b) - f(a)$$

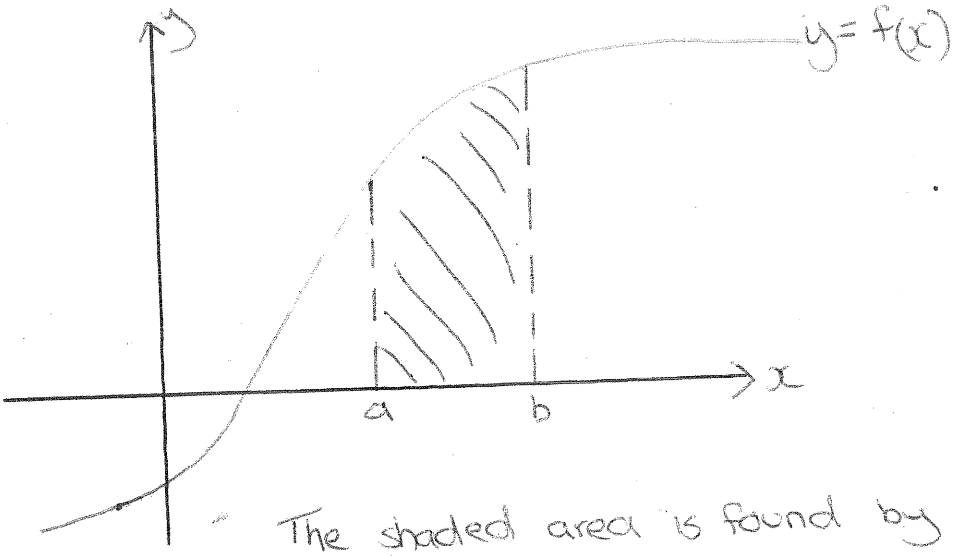
Examples:

1) $\int_1^4 2x.dx = [x^2]_1^4$
$= (4^2) - (1)^2$
$= (16) - (1)$
$= 15$

2) $\int_1^3 (x^2 - 2x).dx = [\frac{x^3}{3} - x^2]_1^3$
$= (9 - 9) - (\frac{1}{3} - 1)$
$= (0) - (-\frac{2}{3})$
$= \frac{2}{3}$

3) $\int_{-1}^2 (5x+1).dx = [\frac{5x^2}{2} + x]_{-1}^2$
$= (10 + 2) - (2.5 + -1)$
$= (12) - (1.5)$
$= 10.5$

Application of integration

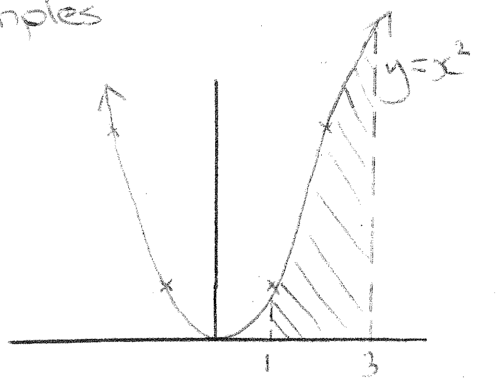


The shaded area is found by evaluating

$$\int_a^b f(x).dx$$

Examples

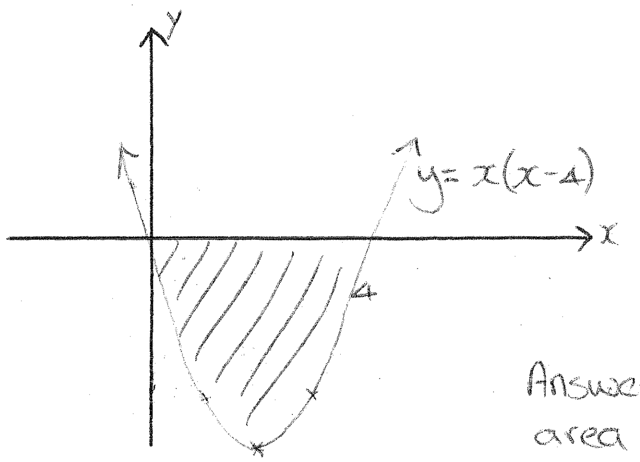
1)



Find the shaded area

$$\text{Area} = \int_1^3 x^2 \cdot dx$$
$$= \left[\frac{x^3}{3}\right]_1^3$$
$$= (9) - (1/3)$$
$$= 8\tfrac{2}{3} \text{ unit}^2$$

2)



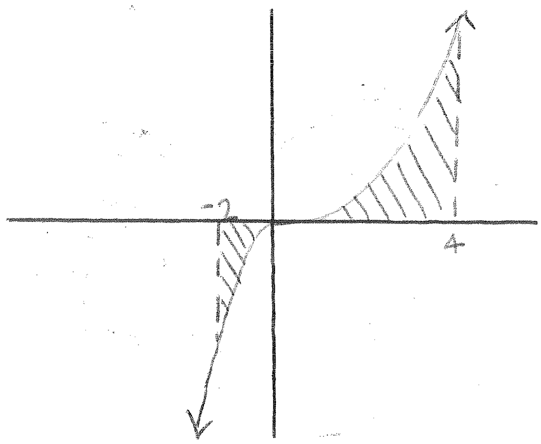
Find the shaded area

$$\int_0^4 (x^2 - 4x) \cdot dx$$
$$= \left[\frac{x^3}{3} - 2x^2\right]_0^4$$
$$= \left(\frac{4^3}{3} - 2(4)^2\right) - (0)$$
$$= {}^{-}10\tfrac{2}{3}$$

Answer is negative because area is below x axis

Area $= 10\tfrac{2}{3}$ units^2

3)



This must be evaluated in two parts.

$$A_1 = \int_{-2}^0 x^3 \cdot dx = \left[\frac{x^4}{4}\right]_{-2}^0$$
$$= (0) - (4)$$
$$= {}^{-}4$$

$$A_2 = \int_0^4 x^3 \cdot dx = \left[\frac{x^4}{4}\right]_0^4$$
$$= (64) - (0)$$
$$= 64$$

Total area $= 4 + 64$
$= 68 \text{unit}^2$

Rate of change

Derivative = rate of change

1) A fire covers 3ha when discovered. It is spreading rapidly and its area after t hours is given by

$$A = 3 + 2t + \frac{1}{2}t^2$$

At what rate is the fire spreading after 4 hours?

$$\frac{dA}{dt} = 2 + t$$

$$\text{Rate} = 2 + 4$$
$$= 6 \text{ ha/hour}$$

3) A water tank is filling at a rate of $60 + 4t$ L/s. Initially the tank had 15000L of water in it. Calculate how long it will take before the tank holds 100000L

Given $\frac{dV}{dt} = 60 + 4t$

$$V = \int (60 + 4t) \cdot dt$$
$$V = 60t + 2t^2 + C$$

Given that $V = 15000$ when $t = 0$

$$15000 = 0 + 0 + C$$
$$\therefore C = 15000$$

Thus $V = 2t^2 + 60t + 15000$

To find when $V = 1000\,000$ L