Achievement Standard 3·6

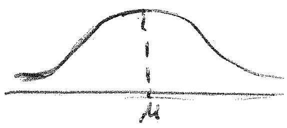
Probability Distributions

The probability distributions that we will study in depth are:

- Normal Distribution
- Binomial Distribution
- Poisson Distribution

Normal Distribution

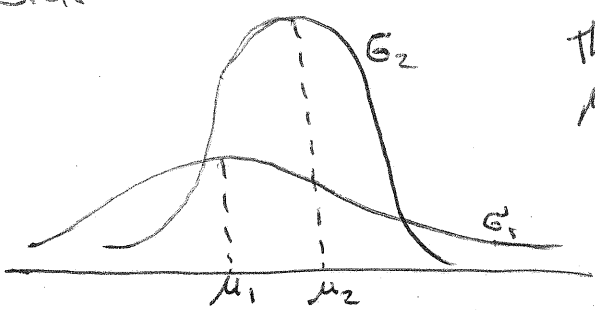
Many continuous distributions, when graphed, have a familiar bell-shaped curve.



Such distributions are said to be normally distribut
Examples of variables which may be normally distributed a

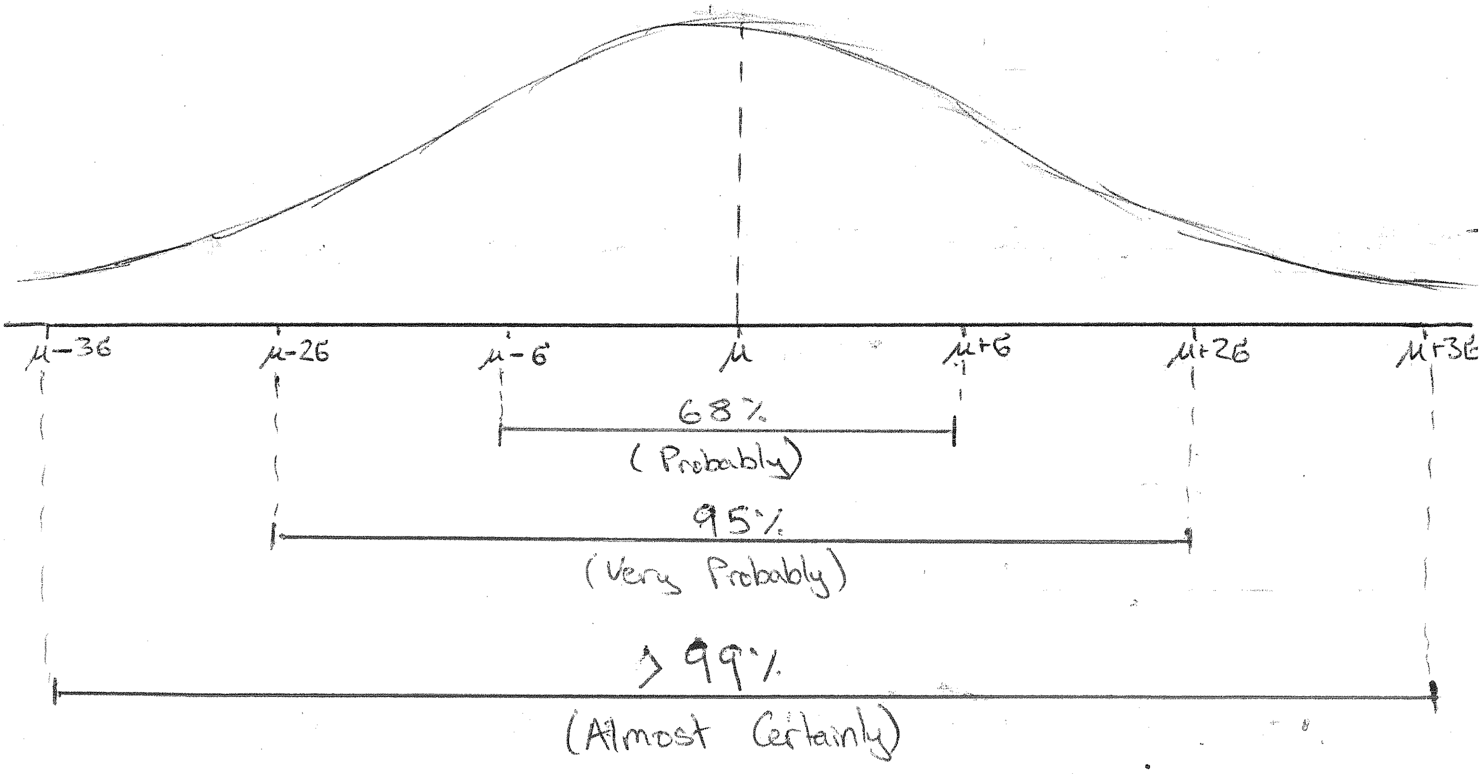
- heights
- weights
- lifetimes

The shape of the curve depends upon the value of the mean & s.d.



The normal curves
$\mu_1 < \mu_2$ & $\sigma_1 > \sigma_2$

Properties of Normal Curve



Standard Normal Curve

All normal distributions X can be transformed to a standard normal distribution Z using the transformation

$$Z = \frac{x - \mu}{\sigma}$$

For a standard normal curve;

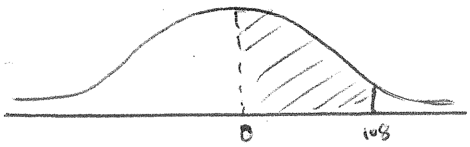
$$\mu_Z = 0$$

$$\sigma_Z^2 \; \& \; \sigma_Z = 1$$

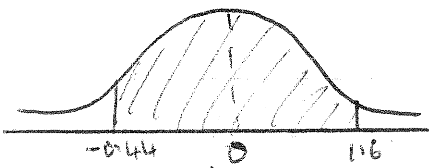
Normal Tables

$\leq$ & $<$ are the same thing

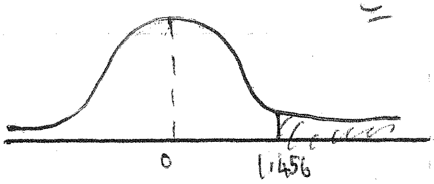
(1) $P(0 < Z < 1{\cdot}8) = 0{\cdot}4641$



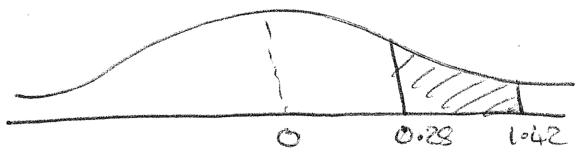
(2) $P(-0{\cdot}44 < Z < 1{\cdot}6) = 0{\cdot}1700 + 0{\cdot}4452 = 0{\cdot}6152.$



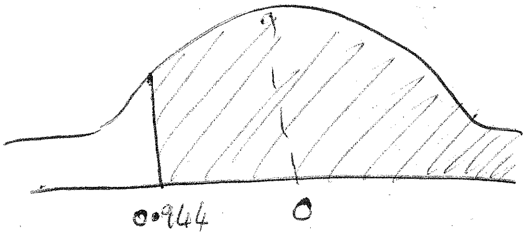
(3) $P(Z > 1{\cdot}456) = 0.5 - 0{\cdot}4273$
$= 0{\cdot}0727$



(4) $P(0.28 < Z < 1.42) = 0.4222 - 0.1103$
$= 0.3119$



(5) $P(Z > -0.944) = 0.5 + 0.3274$
$= 0.8274$



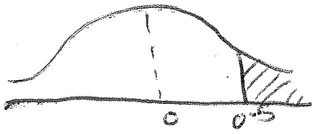
Examples:

(1) the weights of pumpkins grown organically are normally distributed with mean

$\mu = 3.8\text{kg}$ & s.d. $\sigma = 0.4\text{kg}$.

(a) Find the probability that the weight of a pumpkin exceeds 4kg.

$$P(X > 4) = P\left(Z > \frac{4 - 3.8}{0.4}\right)$$
$$= P(Z > 0.5)$$
$$= 0.5 - 0.1915$$
$$= 0.3085$$



(b) Over Page

(b) The market will only accept pumpkins weigh[ing] between 3kg & 4·3kg.

What proportion of pumpkins are accepted.

$$P(3 \leq Z \leq 4.3) = P\left(\frac{3-3.8}{0.4} \leq Z \leq \frac{4.3-3.8}{0.4}\right)$$

$$= P(-2 \leq Z \leq 1.25)$$

$$= 0.4772 + 0.3944$$

$$= 0.8716$$

(c) Pumpkins weighing more than 4·5kg are eligib[le] for prizes. In a harvest of 850 pumpkins, h[ow] many would you expect to win prizes?

$$P(x > 4.5) = P\left(Z > \frac{4.5-3.8}{0.4}\right)$$

$$= P(Z > 1.75)$$

$$= 0.5 - 0.4599$$

$$= 0.0401.$$

Number of pumpkins $= 850 \times 0.0401$

$= 34.085$

$= 34$ pumpkins.

R) The mean speed for cars travelling on a motorway is 96 km/h with a s.d. of 7.3 km/h.

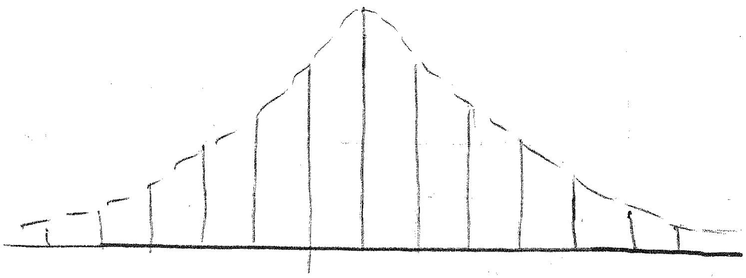
Speeding tickets are given to drivers exceedin 110 km/h.

What proportion of drivers recieve tickets?

$$P(x > 110) = P\left(Z > \frac{110-96}{7.3}\right)$$
$$= P(Z > 1.917808219)$$
$$= P(Z > 1.918)$$
$$= 0.5 - 0.4724$$
$$= 0.0276$$

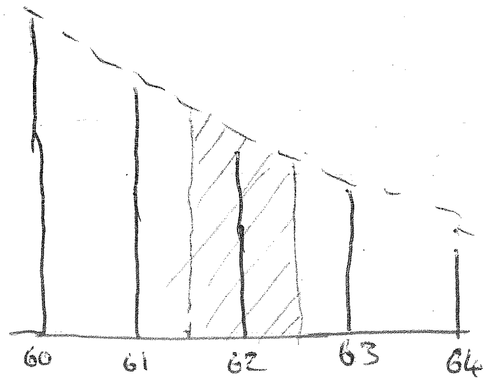
Continuity Corrections

Some discrete distributions, when graphed, have the shape of a normal distribution



eg. I.Q. scores, exam marks
We can use the normal distribution in such cases but must apply a continuity correction.

eg. Exam marks are normally distributed with a mean of 58 & a s.d. of 10.



Find the following

(a) P(Score of 62)

We calculate the area under the curve between 61·5 & 62·5

We only use a C.C. when using a continuous di... on a discrete set of data

P(score of 62) = $P(61 \cdot 5 \leq X < 62 \cdot 5)$

$$= P\left(\frac{61.5-58}{10} \leq Z < \frac{62.5-58}{10}\right)$$

$$= P(0.35 \leq Z < 0.45)$$

$$= 0.1736 - 0.1368 = 0.0368$$

b) P(score at least 55)

$= P(X \geq 54.5)$

$= 0.6368$

c) P(score less than 70)

$= P(X < 69.5)$

$= 0.8749$

Inverse Normal.

This situation arises when the probability is giv & the x value is to be found.

eg (1) The weight of adults are normally distribu with a mean $\mu = 88$kg & a s.d. $\sigma = 12$kg

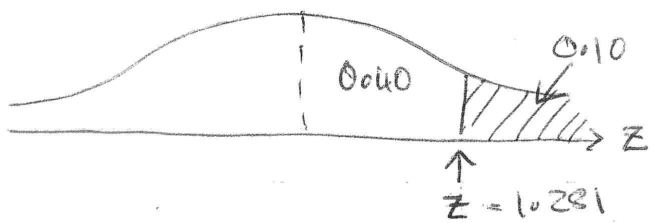
the heaviest 10% of people are classed a being obese

What weight is required to be obese?

Let weight be k.

$P(X > k) = 0{\cdot}10$

$P(Z > \frac{k-88}{12}) = 0{\cdot}10$



$$\frac{k-88}{12} = 1{\cdot}281$$

$k - 88 = 15{\cdot}372$

$k = 103{\cdot}4$ kg.

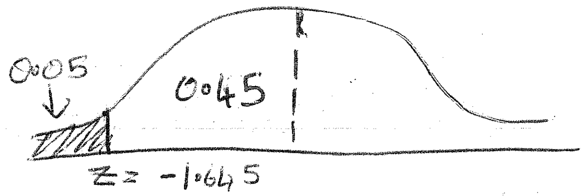
(2) T.V.s have a mean lifetime of 8 years with a s.d. of 1.2 years.

The manufacturers are prepared to replace 5% of them under guarantee.
How long should the guarantee be?

Let guarantee period be k.

$P(x < k) = 0.05$

$P\left(Z < \frac{k-8}{1.2}\right) = 0.05$



$$\frac{k-8}{1.2} = -1.645$$

$k = 6.03$ years.

Harder Problems

(1) A truck carries 2 containers. The truck is lisensed to carry a load of 4500kg.

The weight W of a container is normally distribut with mean $\mu = 2000$kg & s.d. $\sigma = 200$kg.

What is the probability that the truck is overloaded?

We require μ_{w+w} & σ_{w+w}

$$\mu_{w+w} = \mu_w + \mu_w = 4000\text{kg}$$

$$\sigma^2_{w+w} = \sigma^2_w + \sigma^2_w = (200)^2 + (200)^2$$
$$= 80000$$
$$\sigma_{w+w} = \sqrt{80000}$$

$$P(\text{overloaded}) = P(x > 4500)$$
$$= P\left(z > \frac{4500 - 4000}{\sqrt{80000}}\right)$$
$$= P(z > 1.768)$$
$$= 0.5 - 0.4615$$
$$= 0.0385$$

Binomial Distribution

A binomial experiment possesses the following properties:

(1) the experiment consists of n repeated trials

(2) Each trial has 2 possible outcomes, success S or failure F.

(3) In each trial the probability π of a success remains constant

(4) The trials are independent of each other.

The binomial distribution is a discrete distribution with the random variable x being the number of successes recorded in the n trials

ie $x \in \{0, 1, 2, 3, \ldots, n\}$

The probability of getting x successes from n trials is denoted by $b(x, n, \pi)$

$$b(x, n, \pi) = {}^{n}C_{x} \times \pi^{x} \times (1-\pi)^{n-x}$$

e.g. A basketballer is successful with $\frac{3}{5}$ of his free throws.
What is the probability he has 6 successes from his next 8 throws?

$$b(x, n, \pi) = {}^nC_x \times (\pi)^x \times (1-\pi)^{n-x}$$

$$b(6, 8, \tfrac{3}{5}) = {}^8C_6 \times (\tfrac{3}{5})^6 \times (\tfrac{2}{5})^2$$

$$= 28 \times 0.046656 \times 0.16$$

$$= 0.20901888$$

$$= 0.2090$$

Binomial Tables: Tables exist for some values of b(x

eg

(1) $b(3, 10, 0.25) = 0.2503$

(2) $b(7, 10, 0.6) = b(3, 10, 0.4)$
$= 0.2150$

(3) We can also use the tables for inverse problems

If $b(0, 10, \pi) = 0.6$

then using tables, $\pi \approx 0.05$

For a more accurate evaluation of π

$b(0, 10, \pi) = 0.6$

$$^{10}C_0 \times \pi^0 \times (1-\pi)^{10} = 0.6$$

$$1 \times 1 \times (1-\pi)^{10} = 0.6$$

$$(1-\pi) = \sqrt[10]{0.6}$$

$$1-\pi = 0.9502$$

$$\pi = 0.0498$$

Examples (1): The probability of recovering from a rare disease is 0·6
If 10 people have the disease, find the probability that:

a) at least 7 survive

$$b(\text{at least } 7) = \sum_{7}^{10} b(x, 10, 0{\cdot}6)$$

$$= \sum_{3}^{0} b(x, 10, 0{\cdot}4)$$

$$= 0{\cdot}0060 + 0{\cdot}0403 + 0{\cdot}1209 + 0{\cdot}2150$$
$$= 0{\cdot}3822$$

b) less than 5 survive

$$P(\text{less than } 5) = \sum_{0}^{4} b(x, 10, 0{\cdot}6)$$

$$= \sum_{10}^{6} b(x, 10, 0{\cdot}4)$$

$$= 0{\cdot}1115 + 0{\cdot}0425 + 0{\cdot}0106 + 0{\cdot}0016 + 0{\cdot}00$$
$$= 0{\cdot}1663$$

(2) A manufacturer of washing machines knows that 18% of them will develop a fault in the first 3 years.
Find the probability that in a batch of 20 washing machines:

a) Fewer than 3 will develop a fault:
We can use graphics calculator, + select cumulative binomial (bc

The calculator finds $\sum_{0}^{x} b(x, n, \pi)$
The values of x, n, π must be entered.

$$\sum_{0}^{2} b(x, 20, 0{\cdot}18)$$

$= 0{\cdot}2748$

b) between 2 & 4 inclusive will develop a faul

We require $\sum_{2}^{4} b(x, 20, 0{\cdot}18)$

$$= \sum_{0}^{4} b(x, 20, 0{\cdot}18) - \sum_{0}^{1} b(x, 20, 0{\cdot}18)$$

$= 0{\cdot}7151 - 0{\cdot}1018$

$= 0{\cdot}6133$

Mean & Standard Deviation (Binomial)

The random variable X representing the number of successes in a binomial expt. consisting of n trials has

mean $\mu_x = n\pi$

s.d. $\sigma_x = \sqrt{n\pi(1-\pi)}$

Example: A multichoice test has 80 questions with 4 possible answers for each question, only 1 of which is correct.

Simon has no idea how to do any of them so he guesses.
Find the mean & s.d. of the number of questions he can expect to get correct.

mean $\mu = n\pi = 80 \times \frac{1}{4} = 20$

s.d. $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{80 \times \frac{1}{4} \times \frac{3}{4}} = \sqrt{15}$

Poisson Distribution

Experiments yielding a numerical value of a random variable X. representing the number of events occuring in a given time interval, or in a given region; are called Poisson experiments

Examples are:

- number of wrong numbers dialled per day

- " " hailstorms per year

- " " misprints per page

A poisson distribution has the following properties

1- no upper limit to the number of events that can occur
2- events cannot occur simultaneously
3- events occur randomly
4- events occur independantly of each other
5- events occur at a constant rate on average.

The only parameter of the poisson distribution is λ repr the mean number of events expected to occur

The poisson probability of x events occuring whe λ were expected is.

$$P(x, \lambda) = \frac{e^{-\lambda} \times \lambda^{x}}{x!}$$

e.g.(1) Suppose on average Dunedin has 2.8 snow days per year

Find the probability that a particular year will have 4 snow days

Here $\lambda = 2.8$

Require $p(4, 2.8) = \dfrac{e^{-2.8} \times 2.8^4}{4!}$

$= 0.1557$

(2) A receptionist finds that on average she receives 8 malicious phone calls per hour.
Find the probability that:

(a) she receives 3 in a 1 hour period
$p(3,8)$ can be evaluated using a graphics calculator.
We require an individual term so select Ppd.
$p(3,8) = 0.0286$

(b) she receives 3 in a 15 minute period.

Here $\lambda = \frac{1}{4}(8) = 2$

Require $p(3,2) = 0.1804$

(c) She receives at least 4 malicious phone calls in a 15 minute period.
We require $\sum_{4}^{\infty} p(x,2) = 1 - \sum_{0}^{3} p(x,2)$

We can use a G.C. & select Pcd since we require a cumulative poisson probability.

$\sum_{4}^{\infty} p(x,2) = 1 - 0.85712 = 0.1429$

Inverse Problems

These involve being given a probability and being asked to find the poisson parameter λ.

We must know the probability of no events occurring

e.g. In samples of milk taken from a tanker, 40% were found to contain no bacterial spore

Calculate the mean number of spore per sample, and hence the probability that the sample contains exactly 2 spore.

We are asked to find the mean, λ

Given $p(0, \lambda) = 0.4$

$$\frac{e^{-\lambda} \times \lambda^0}{0!} = 0.4$$

$$e^{-\lambda} = 0.4$$

$$\ln(e^{-\lambda}) = \ln 0.4$$
$$-\lambda = \ln 0.4$$
$$\lambda = 0.92$$

$$P(2 \text{ spore}) = p(2, 0.92)$$
$$= 0.1687$$